HOMEWORK 10
MATH 430, SPRING 2014

Recall the result from last homework, that if $\mathfrak{B} \models PA$, then it is an end extension of $\mathfrak{A}$ (the standard model). Use this for the following problem.

Problem 1. (1) *Suppose that ϕ is a Σ_1 formula and $\mathfrak{A} \models \phi$. Show that $PA \models \phi$. I.e. show that any $\mathfrak{B} \models PA$ is a model of ϕ.*
(2) *Suppose that ϕ is a Π_1 formula and for some model $\mathfrak{B}$ of PA, $\mathfrak{B} \models \phi$. Show that $\mathfrak{A} \models \phi$.*

For $i < n$, define the projection $P_i^n : \mathbb{N}^k \to \mathbb{N}$ to be $P_i^n(x_0, ..., x_{n-1}) = x_i$. The **primitive recursive functions** are functions $f : \mathbb{N}^k \to \mathbb{N}$ that are built up from the constant function $x \mapsto 0$, successor, and projection, by applying composition and the primitive recursive operation:

- $f(0, x_1, ..., x_{k-1}) = g(x_1, ..., x_{k-1})$,
- $f(S(y), x_1, ..., x_{k-1}) = h(y, f(y, x_1, ..., x_{k-1}), x_1, ..., x_{k-1})$.

where g and h are primitive recursive.

Problem 2. *Show that addition is primitive recursive. I.e. you have to show that the function $(x, y) \mapsto x + y$ can by written as above.*

For a function $f : \mathbb{N}^k \to \mathbb{N}$, ϕ_f will denote the formula such that

$$\mathfrak{A} \models \phi_f[a_0, ..., a_{k-1}, b] \text{ iff } f(a_0, ..., a_{k-1}) = b.$$

For example, if f is the addition function, then $\phi_f(x, y, z)$ is the formula $x + y = z$. Note that for each of the $x \mapsto 0$, successor, projection, addition, multiplication, the corresponding formula is atomic and therefore Δ_0.

Problem 3. *Suppose that f is defined by primitive recursion from functions g, h as above and suppose that ϕ_g, ϕ_h are Δ_1. Write down ϕ_f in terms of ϕ_g, ϕ_h and show it is also Δ_1.*

Problem 4. *Suppose that $f : \mathbb{N}^k \to \mathbb{N}$ and that ϕ_f is Σ_1. Show that ϕ_f is also equivalent to a Π_1 formula, and therefore Δ_1.*
Hint: It is enough to show that $\neg\phi_f$ is equivalent to Σ_1. Then use that negations of Σ_1 formulas are equivalent to Π_1 formulas.

FYI: The **partial recursive functions** recursive functions are built up from primitive recursive functions by including "minimization": if g is recursive, then

$$f(x_0, ..., x_{n+1}) = \text{ the least } y \text{ such that } g(y, x_0, ..., x_{n+1}) = 0$$

is partial recursive. I.e. its domain is a subset of $\mathbb{N}^n$. If the domain is all of $\mathbb{N}^n$, then f is total recursive. One can show that:

- f is partial recursive iff ϕ_f is Σ_1;
- f is total recursive (or just recursive) iff ϕ_f is Δ_1;